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Coherent evolution in a multiply connected space in the presence of a fractional magnetic flux: the back-reaction effect

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Abstract. Electric charges on a circle in the presence of a fractional magnetic flux $(2\pi/e)(s/q)$ are considered. A winding operator $\hat{w}$ and a flux operator $\hat{s}$ are defined and the corresponding $Z(q) \times Z(q)$ phase space is shown to describe the global properties of the system. The Heisenberg–Weyl group of discrete displacements and the $SL(2, Z(q))$ group of discrete Bogoliubov transformations in the s–w phase space, are studied. They describe coherent evolution of the system, going beyond the external field approximation and taking into account the back-reaction of the electric charge on the magnetic flux. When q is the power of a prime, $Z(q)$ is a Galois field and stronger results can be proved.

1. Introduction

Quantum mechanics in multiply connected spaces has been studied by various authors [1–5]. It has been inspired by the Aharanov–Bohm effect [6] and provides the theoretical background for the understanding of many phenomena (e.g. magnetoresistance oscillations in small rings [7]).

An electric charge is moving on a circle and a magnetostatic flux ϕ is threading the centre of the circle, in the perpendicular direction. The wavefunction $R(x)$ obeys the quasi-periodic boundary condition

$$R(x + 2\pi) = R(x) \exp(\mathrm{i}\theta) \tag{1}$$

where $\theta = e\phi$. We consider the case of magnetic flux which is a rational multiple of the flux quantum (fluxon):

$$\phi = \frac{2\pi}{e} \frac{s}{q} \tag{2}$$

where s and q are integers. In this case equation (1) becomes

$$R(x + 2\pi) = R(x) \exp\left(\mathrm{i}\frac{2\pi s}{q}\right). \tag{3}$$

The Hilbert space of these functions for a fixed s, is a subspace of periodic functions with period $2\pi q$:

$$R(x + 2\pi q) = R(x). \tag{4}$$

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We work in the bigger Hilbert space of equation (4) for two reasons. The first is that any observation of the paths [8–10] can easily relax the restrictive condition (3). With such measurements (and we shall give explicitly the relevant operators in equations (39) and (40)) the winding number acquires a certain value and the phase θ becomes uncertain. In this case the wavefunction belongs to the bigger Hilbert space of equation (4). The second reason is related to the fact that most of the existing literature tacitly assumes the external field approximation, in which any back-reaction from the electric charge on the magnetic flux is neglected. Within this approximation the value of the flux is fixed and therefore s in (3) has a fixed value. This is a reasonable approximation in a certain limit of the parameters of the experiment; but there is an opposite limit where the back-reaction is significant. In this paper we are interested in the latter case and we want to go beyond the external field approximation. The magnetic flux ϕ is no longer constant because the back reaction creates some extra magnetic flux. Consequently the Hilbert space of (3) is too restrictive, and we need to work on the bigger Hilbert space of (4). Within this Hilbert space we introduce two bases $v(x; N, s)$ and $u(x; N, w)$ which are related to each other through a finite Fourier transform. s and w are integers in $Z(q)$ (the integers modulo q) and are dual to each other in the Fourier transform sense. s is effectively the magnetic flux (with a factor $2\pi/qe$) and w is the winding number. The (s, w) take values in $Z(q) \times Z(q)$ which can be viewed as a kind of phase space describing the global properties of the system. In this phase space we introduce techniques from the theory of finite quantum systems (e.g. [11–15]).

We define the winding operator $\hat{w}$ and the flux operator $\hat{s}$ and use them to study displacement operators and the corresponding finite Heisenberg–Weyl group. We also study $SL(2, Z(q))$ transformations in this context. All these transformations evolve the system coherently in the s–w phase space. In order to appreciate this we first point out that the analogue of these transformations for a harmonic oscillator are the displacement operators

$$D(a_1, a_2) = \exp[\mathrm{i}(a_1 x + a_2 p)] \tag{5}$$

associated with the Heisenberg–Weyl group and the squeezing operators

$$S(b_1, b_2, b_3) = \exp[\mathrm{i}(b_1 x^2 + b_2 p^2 + b_3 x p)] \tag{6}$$

associated with the $SL(2, R)$ group. The study of these transformations provides an understanding of the evolution of a system with a general quadratic Hamiltonian

$$H = b_1 x^2 + b_2 p^2 + b_3 x p + a_1 x + a_2 p. \tag{7}$$

In the case of a finite phase space like the s–w phase space that we study here, the displacement operators are again exponentials of linear functions of s and w, and the $SL(2, Z(q))$ transformations are quadratic functions of s and w. There is a big difference with the harmonic oscillator case however, which is that the coefficients take discrete values here. Consequently the Heisenberg–Weyl and $SL(2, Z(q))$ transformations in our context describe the evolution of systems with quadratic Hamiltonians

$$H = b_1 s^2 + b_2 w^2 + b_3 s w + a_1 s + a_2 w \tag{8}$$

where the coefficients take discrete values, and for a discrete set of times (stroboscopic evolution in the spirit of [15]). We call this coherent evolution in the sense that there is a group structure; the product of two such evolution operators is another evolution operator in the same group, for every evolution operator there exists an inverse, etc. Clearly a good understanding of coherent evolution is a pre-requisite for the understanding of other more complicated types of evolution (e.g. at times other than the stroboscopic ones or with other more complicated Hamiltonians etc).

The Hamiltonian (8) has a nice physical interpretation, in spite of the restriction about the coefficients taking discrete values. Our system can be viewed as two coupled solenoids: one is the electric charge on the circle (with flux proportional to w) and the other is the solenoid that creates the magnetic flux threading the circle (with flux proportional to s). The first two terms of the Hamiltonian (8) are the self-energies of these two solenoids and the third term is the mutual energy. The fourth and fifth terms are linear terms associated with classical sources. For example, if the flux ϕ has a classical part, this will produce the linear terms in s and w. The Hamiltonian (8) shows clearly that we study the full interaction between the electric charge and the magnetic flux tube, going beyond the external field approximation and taking into account the back-reaction of the electric charge on the magnetic flux tube. In particular, the $SL(2, Z(q))$ transformations associated with the quadratic part of this Hamiltonian are essential for the description of the interaction beyond the external field approximation.

The work is related to areas such as anyons [16], Abelian Chern–Simons theories [17], quantum groups, etc. Composites made from one electric charge and one magnetic flux tube (anyons), exhibit fractional statistics and have been studied extensively in the literature. The relation between anyons and quantum groups has been studied in [18]. It should be clear, however, that in anyons we consider many charge-flux tube composites and study their statistics, their thermodynamics, etc. Here we consider only one tube of magnetic flux and electric charges that go around it, and describe this system taking into account the back-reaction of the electric charges on the magnetic flux tube. In this sense, our work is directly applicable to the Aharanov–Bohm type of experiments in which the parameters are such that the back-reaction from the electric charges on the external magnetic flux cannot be neglected.

2. The Hilbert space of periodic functions with period $2\pi q$

We consider the Hilbert space $H(q)$ of all periodic functions with period $2\pi q$ (where q is an integer):

$$R(x + 2\pi q) = R(x). \tag{9}$$

Their Fourier expansion is given by

$$R(x) = \sum_{M=-\infty}^{\infty} a(M) \exp\left(\mathrm{i}\frac{Mx}{q}\right). \tag{10}$$

Let N and s be the integer part and remainder correspondingly of the division M/q:

$$\frac{M}{q} = N + \frac{s}{q}. \tag{11}$$

We rewrite equation (10)

$$R(x) = \sum_{s=0}^{q-1} \psi(x; s) \tag{12}$$

$$\psi(x; s) = \sum_{N=-\infty}^{\infty} a(N; s) v(x; N, s) \tag{13}$$

$$v(x; N, s) = \exp\left[\mathrm{i}\left(N + \frac{s}{q}\right) x\right] \tag{14}$$

where s is an integer in $Z(q)$ (the integers module q). It is easily seen that

$$\int_0^{2\pi q} v(x; N_1, s_1) v^*(x; N_2, s_2) \frac{\mathrm{d}x}{2\pi q} = \delta(N_1, N_2)\delta(s_1, s_2) \tag{15}$$

$$\frac{1}{q} \sum_{N,s} v(x; N, s) v^*(y; N, s) = \delta(x - y) \tag{16}$$

where $\delta(N_1, N_2)$ denotes Kronecker delta. The coefficients $a(N; s)$ are given by:

$$a(N, s) = \int_0^{2\pi q} R(x) v^*(x; N, s) \frac{\mathrm{d}x}{2\pi q}. \tag{17}$$

We see

$$\omega = \exp\left[\mathrm{i}\frac{2\pi}{q}\right] \tag{18}$$

and we use the shorthand notation $\omega(x)$ for ω^x . The following formula will be useful,

$$\frac{1}{q} \sum_{m=0}^{q-1} \omega[m(\alpha - \beta)] = \delta(\alpha, \beta) \tag{19}$$

where α and β are integers in $Z(q)$. More generally, we shall use the sum

$$d_0(x) = \frac{1}{q} \sum_{m=0}^{q-1} \omega(mx) = \frac{1}{q} U_{q-1}\left[\cos\left(\frac{\pi x}{q}\right)\right] \tag{20}$$

where U_{q-1} are Chebyshev polynomials of the second kind. In the appendix of [14] we have studied these functions and their derivatives, because they appear very often in the study of finite quantum systems. In fact they are the analogues of the delta function and its derivatives in infinite quantum systems.

We can now prove

$$v(x + 2\pi; N, s) = v(x; N, s)\omega(s) \tag{21}$$

$$\psi(x + 2\pi, s) = \psi(x, s)\omega(s) \tag{22}$$

$$\psi(x, s) = \frac{1}{q} \sum_{m=0}^{q-1} R(x + 2\pi m)\omega(-sm). \tag{23}$$

We next perform a finite Fourier transform on the functions $v(x; N, s)$ with respect to the variable s:

$$u(x; N, w) = q^{-1/2} \sum_{s=0}^{q-1} \omega(sw) v(x; N, s) = q^{1/2} \exp(\mathrm{i}Nx) d_0\left(\frac{x}{2\pi} + w\right). \tag{24}$$

Note that

$$u(x + 2\pi; N, w) = u(x; N, w + 1) \tag{25}$$

$$\int_0^{2\pi q} u(x; N_1, w_1) u^*(x; N_2, w_2) \frac{\mathrm{d}x}{2\pi q} = \delta(N_1, N_2)\delta(w_1, w_2) \tag{26}$$

$$\frac{1}{q} \sum_{N,w} u(x; N, w) u^*(y; N, w) = \delta(x - y). \tag{27}$$

Using them we get an alternative expansion for $R(x)$:

$$R(x) = \sum_{w=0}^{q-1} \chi(x, w) \tag{28}$$

$$\chi(x, w) = \sum_{N=-\infty}^{\infty} b(N, w) u(x; N, w) \tag{29}$$

$$b(N, w) = \int_0^{2\pi q} \psi(x) u^*(x; N, w) \frac{\mathrm{d}x}{2\pi q}. \tag{30}$$

It is easily seen that

$$\chi(x + 2\pi, w) = \chi(x, w) \tag{31}$$

and that $a(N, s)$ and $b(N, w)$ are related through the finite Fourier transform:

$$b(N, w) = q^{-1/2} \sum_{s=0}^{q-1} a(N, s)\omega(-sw) \tag{32}$$

$$a(N, s) = q^{-1/2} \sum_{s=0}^{q-1} b(N, w)\omega(sw). \tag{33}$$

We now introduce the Hilbert spaces $H(q; N)$ spanned by the functions

$$H(q; N) = \{v(x; N, s); s \in Z(q)\} \tag{34}$$

the Hilbert spaces $H(q; s)$ spanned by the functions

$$H(q; s) = \{v(x; N, s); N \in Z\} \tag{35}$$

and the Hilbert spaces $\tilde{H}(q; w)$ spanned by the functions

$$\tilde{H}(q; w) = \{u(x; N, w); N \in Z\}. \tag{36}$$

It is clear that

$$H(q) = \sum_{N=-\infty}^{\infty} H(q; N) = \sum_{s=0}^{q-1} H(q; s) = \sum_{w=0}^{q-1} \tilde{H}(q; w) \tag{37}$$

where the summation denotes the direct sum. $H(q; N)$ are q-dimensional Hilbert spaces which for the various values of N form a 'ladder' of Hilbert spaces. In each $H(q; N)$ we have defined the v-basis and the u-basis, which are related to each other through a finite Fourier transform. $v(x; N, s)$ with fixed s and variable N span the infinite-dimensional Hilbert space $H(q; s)$. $u(x; N, s)$ with fixed s and variable N span the infinite-dimensional Hilbert space $\tilde{H}(q, w)$. A similar decomposition of the Hilbert space has been used in a different context in [19]. Note, however, that there we had the harmonic oscillator Hilbert space, while here we consider the Hilbert space of periodic functions of equation (9). The interpretation is also different in the two cases.

3. The *w–s* phase space: a discretized torus describing the global properties of the system

We consider a quantum mechanical particle on a circle ($0 \leqslant x < 2\pi$). A magnetic flux ϕ, perpendicular to the plane of the circle, threads it through its centre. The wavefunction $R(x)$ of the particle is known to obey the quasi-periodic boundary condition of equation (1) with $\theta = e\phi$. Since the phase θ enters in the exponential, it is defined mod (2π) and the flux ϕ is defined mod $(2\pi/e)$.

We next consider the case of a fractional flux given by equation (2), where q is an integer, and s belongs to $Z(q)$. In this case equation (1) becomes

$$R(x+2\pi) = R(x)\omega(s). \tag{38}$$

Clearly $R(x)$ is a periodic function with period $2\pi q$. In fact it belongs to the Hilbert space $H(q; s)$ (equation (35)). We have already explained in the introduction that for our purposes we need to consider the larger Hilbert space $H(q)$, of all periodic functions with period $2\pi q$.

When an experiment is performed there are two extreme cases. In one, the phase θ has a certain value ($\theta = e\phi = 2\pi s/q$), while the dual (in the finite Fourier transform sense) variable w, which can be interpreted as winding number, has large uncertainty. This is realized in the Aharanov–Bohm type of experiments (e.g. [6]), where the paths of the particles are not observed and we get interference. In the second extreme case, we have large uncertainty in θ and a certain value of w. This is realized in an Aharanov–Bohm type of geometry, where the paths of the particles are observed in order to determine their winding number. The phase θ is now disturbed by the measuring apparatus, and the interference is destroyed. This has been discussed by Feynman [8], Furry and Ramsey [9] etc. More recently it has also been discussed in the context of 'which path' experiments in [10]. In our context, observation of the paths can be done with measurement (Hermitian) operators whose eigenvectors are $u(x; N, w)$:

$$\Lambda = \sum_{w=0}^{q-1} w\pi_w \tag{39}$$

$$\pi_w = \sum_{N=-\infty}^{\infty} u(x; N, w)u(y; N, w). \tag{40}$$

They disturb the phase θ but still leave it within $Z(q)$. Apart from the above two extreme cases, there are in between situations where we have some uncertainty in the phase θ , some uncertainty in the winding number w, and partial destruction of the interference. In the first extreme case, where θ has the fixed value $\theta = e\phi = 2\pi s/q$, the wavefunctions belong in the Hilbert space $H(q, s)$. In the second extreme case, where the paths are observed and the winding number has a certain value, the wavefunctions belong in the Hilbert space $\tilde{H}(q; w)$. In other intermediate cases, the wavefunction can be anywhere in $H(q)$. It is therefore clear that for a general study of the problem we need to consider the whole $H(q)$. Any restriction into a smaller space like $H(q, s)$ or $\tilde{H}(q; w)$ is appropriate only for special cases of the phenomenon that we study.

An additional, and for our purposes more important, reason for working within $H(q)$ is that we go beyond the external field approximation and take into account the back-reaction of the electric charges on the magnetic flux. In this case it is clearly not sufficient to work within a certain $H(q; s)$ with a fixed s, because this restricts the value of the flux into $(2\pi/e)(s/q)$. We need the bigger Hilbert space $H(q)$ that contains all $H(q; s)$ for all values of s in $Z(q)$, so that the electric charges on the circle can increase or decrease the value of the flux.

So we consider wavefunctions $R(x)$ which are periodic with period $2\pi q$. The position of the particle is characterized by the pair (x, w), where $0 \leqslant x < 2\pi$ and w is the winding number. In the covering space of the circle, which is a real line, the position of the particle is $x+2\pi w$. The momentum of the particle is seen in equations (10) and (11) to be $N+(s/q)$

and is characterized by the pair $(N; s)$. The pair (x, w) is dual (in the Fourier transform sense) to the pair (N, s). x takes values in a circle, its dual N is an integer, and the corresponding $S \times Z$ is the usual phase space of the system (describing local properties). w takes values in $Z(q)$, its dual s also takes values in $Z(q)$ and the corresponding discretized torus

$$T = Z(q) \times Z(q) \tag{41}$$

can be viewed as a phase space associated with the global properties of the system. In this paper we apply various phase-space techniques on the discretized torus T and discuss their physical interpretation.

4. The Heisenberg–Weyl group: discrete displacements

The winding operator $\hat{w}$ is defined to be the operator which has $\chi(x, w)$ as eigenvectors and the winding numbers w as eigenvalues:

$$\hat{w}\chi(x, w) = w\chi(x, w). \tag{42}$$

It can be realized as the integral transform

$$\hat{w} f(x) = \int w(x, y) f(y)\,\mathrm{d}y \tag{43}$$

$$w(x, y) = \sum_{w=0}^{q-1} w\chi(x, w)\chi^*(y, w) \tag{44}$$

where $f(x)$ is an arbitrary function in $H(q)$. Similarly the flux operator $\hat{s}$ is defined to be the operator which has $\psi(x, s)$ as eigenvectors and s as eigenvalues

$$\hat{s}\psi(x, s) = s\psi(x, s). \tag{45}$$

It can be realized as the integral transform

$$\hat{s} f(x) = \int s(x, y) f(y)\,\mathrm{d}y \tag{46}$$

$$s(x, y) = \sum_{s=0}^{q-1} s\psi(x, s)\psi^*(y, s). \tag{47}$$

Since both w and s are defined modulo q, the operators $\hat{w}$ and $\hat{s}$ are defined modulo $q\mathbf{1}$.

Displacement operators in the w-direction are defined as

$$E = \exp\left(-\mathrm{i}\frac{2\pi}{q}\hat{s}\right). \tag{48}$$

It is easy to prove that

$$E\chi(x, w) = \chi(x, w + 1) \tag{49}$$

$$E\psi(x, s) = \omega(-s)\psi(x, s). \tag{50}$$

Similarly, we define the displacement operators in the s-direction as

$$F = \exp\left(\mathrm{i}\frac{2\pi}{q}\hat{w}\right). \tag{51}$$

It is easy to prove that

$$F\chi(x, w) = \omega(w)\chi(x, w) \tag{52}$$

$$F\psi(x, s) = \psi(x, s + 1). \tag{53}$$

Both s and w are defined modulo q and therefore

$$E^q = F^q = \mathbf{1}. \tag{54}$$

It is also easy to show that

$$FE = EF\omega. \tag{55}$$

It is now clear that the operators

$$D(\alpha, \beta) = E^\alpha F^\beta \tag{56}$$

where α and β are integers in $Z(q)$, perform discrete displacements in the s–w phase space. They form a finite Heisenberg–Weyl group similar to that studied in a different context in [11–14]. Acting with $D(\alpha, \beta)$ on the general wavefunction

$$R(x) = \sum_{s=0}^{q-1} \psi(x, s) = \sum_{w=0}^{q-1} \chi(x, w) \tag{57}$$

we get

$$D(\alpha, \beta)R(x) = \sum_{s=0}^{q-1} \omega(-\alpha(s+\beta))\psi(x, s+\beta) = \sum_{w=0}^{q-1} \omega(\beta w)\chi(x, w+\alpha). \tag{58}$$

We emphasize that $\hat{s}$ and $\hat{w}$ are not generators of the Heisenberg–Weyl group, which in the present context is discrete. Unlike the harmonic oscillator case, where $\hat{x}$, $\hat{p}$, $\mathbf{1}$, form the Lie algebra corresponding to the Heisenberg–Weyl group and the commutator $[\hat{x}, \hat{p}] = \mathrm{i}\mathbf{1}$ is very important, here the $\hat{s}$ and $\hat{w}$ do not form a Lie algebra and the commutator $[\hat{s}, \hat{w}]$ is not so important. However, for completeness we have calculated its value (in a different context) in [13].

It is clear that the problem considered in this paper reduces to quantum mechanics of finite quantum systems, and the whole formalism developed in that area [11–14] is applicable here. In the next section we discuss how the back-reaction problem is related to discrete Bogoliubov transformations and the $SL(2, Z(q))$ group.

5. The $SL(2, Z(q))$ group: discrete Bogoliubov transformations

In this section we consider transformations that leave the Heisenberg–Weyl group of equations (54) and (55) invariant. Let

$$E' = E^\alpha F^\beta \tag{59}$$

$$F' = E^\gamma F^\delta \tag{60}$$

where α, β, γ and δ are integers in $Z(q)$ such that

$$\alpha\delta - \beta\gamma = 1 \ (\mathrm{mod}\, q). \tag{61}$$

The physical interpretation of these transformations in the present context is that change in the winding number by E^α is associated with simultaneous change in the magnetic flux by F^β. This is related to the diagonalization of the Hamiltonian (8), and the inclusion of the back-reaction. It is easy to show that

$$(E')^q = (F')^q = \mathbf{1} \tag{62}$$

$$F'E' = E'F'\omega. \tag{63}$$

These transformations form the $SL(2, Z(q))$ group, which we have studied in a different context in [14]. There we have constructed explicitly unitary operators U which give

$$UEU^\dagger = E^\alpha F^\beta \tag{64}$$

$$UFU^\dagger = E^\gamma F^\delta. \tag{65}$$

Here we present a different expression for U in terms of exponentials of quadratic functions in s and w. This is important in the present context, because we interpret these operators as stroboscopic evolution operators associated with certain Hamiltonians quadratic in s and w. We consider the unitary operators

$$U_1 = \exp\left(\mathrm{i}\frac{\pi}{q} w^2\right) \tag{66}$$

$$U_2 = \exp\left(\mathrm{i}\frac{\pi}{q} s^2\right) \tag{67}$$

and show that for λ and μ in $Z(q)$

$$U_1^\lambda E^\mu U_1^{-\lambda} = E^\mu F^{\lambda\mu} \omega(\tfrac{1}{2}\lambda\mu^2) \tag{68}$$

$$[U_1, F] = 0 \tag{69}$$

$$U_2^\lambda F^\mu U_2^{-\lambda} = E^{-\lambda\mu} F^\mu \omega(-\tfrac{1}{2}\lambda\mu^2) \tag{70}$$

$$[U_2, E] = 0. \tag{71}$$

Equation (68) can be proved by acting with these operators on the function $\chi(x, w)$ and using equations (49) and (52). Equation (70) can be proved by acting with these operators on the functions $\psi(x, s)$ and using equations (50) and (53). Combining the above equations, we show that the operators

$$U = U_1^\lambda U_2^\mu U_1^\nu \tag{72}$$

lead to the transformations

$$UEU^\dagger = E^\alpha F^\beta \omega(\epsilon) = E' \omega(\epsilon) \tag{73}$$

$$UFU^\dagger = E^\gamma F^\delta \omega(\eta) = F' \omega(\eta) \tag{74}$$

where

$$\alpha = 1 - \mu\nu \tag{75}$$

$$\beta = \lambda - \lambda\mu\nu + \nu \tag{76}$$

$$\gamma = -\mu \tag{77}$$

$$\delta = 1 - \lambda\mu \tag{78}$$

$$\epsilon = \tfrac{1}{2}\lambda(1 - \mu\nu)^2 + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu\nu^2 \tag{79}$$

$$\eta = \tfrac{1}{2}\lambda\mu^2 - \tfrac{1}{2}\mu. \tag{80}$$

The operator E, which increases the winding number, transforms into $E' = E^\alpha F^\beta$ where the displacement in w is accompanied by a displacement in s. In other words, as the winding number increases by E^α, extra flux is created by F^β, which is added to the external flux. This is precisely the inclusion of the back-reaction. A similar argument can be given for F which becomes $F' = E^\gamma F^\delta$.

Acting with the operators U on both sides of w and s we obtain new operators w' and s':

$$w' = UwU^\dagger \tag{81}$$

$$s' = UsU^\dagger. \tag{82}$$

We call these transformations discrete Bogoliubov transformations. In the harmonic oscillator case the analogue of these transformations leads to new position and momentum which are linear combinations of the original ones:

$$x' = ax + bp \tag{83}$$

$$p' = cx + dp \tag{84}$$

$$ad - bc = 1. \tag{85}$$

In equations (81) and (82), we cannot obtain linear combinations similar to (83) and (84). w' and s' can be interpreted as operators along different lines in phase space, so that in w', a change in w is intimately connected with some change in s and also in s' a change in s is intimately connected with some change in w. This is precisely the inclusion of the back-reaction.

We have explained in [14] that $Z(q)$ is, in general, a commutative ring with a unity. If and only if q is a power of a prime p

$$q = p^m \tag{86}$$

$Z(q)$ is Galois field. Stronger results can be proved in the Galois case. For example, in the transformations (59) and (60) due to constraint (61) α, β, γ, and δ, are not all independent. The question then rises whether, for a given triplet (α, β, γ), there exists δ which satisfies (61). In the non-Galois case the answer is that there might or might not exist δ which satisfies (61), and this is clearly a weak statement. In the Galois case, due to the existence of inverses, the answer is affirmative and we have

$$\delta = \alpha^{-1}(1 + \beta\gamma). \tag{87}$$

The phase space $Z(q) \times Z(q)$ in the non-Galois case is a set of points with no geometrical structure. In the Galois case it is a finite geometry [20]. A finite geometry has a finite number of points and a finite number of lines. The $SL(2, Z(q))$ transformations rotate the s and w axes into new s' and w' axes and they can be viewed as 'discrete simplectic transformations' or 'discrete Bogoliubov transformations' in this geometry. Although they are discrete, they are performed on a geometry. Consequently, they have a clear physical interpretation and the results are equally strong with the continuous (harmonic oscillator) case of equations (83) and (84). All these nice properties are intimately related with the existence of inverses in $Z(q)$ in the Galois case. In the non-Galois case $Z(q) \times Z(q)$ is not a finite geometry, and the concepts of s and w axes and rotations into new s' and w' axes are weak. This is intimately related to the fact that in the non-Galois case the inverses in $Z(q)$ do not necessarily exist.

6. Discussion

We have considered an electrical charge on a circle in the presence of an external fractional magnetic flux $(2\pi/e)(s/q)$. The objective has been to go beyond the external field approximation and take into account the back-reaction from the electric charge on the magnetic flux. We found that the Hilbert space $H(q; s)$ of functions that obey equation (3) is too restrictive, because it forces the flux to have the value imposed externally and does not allow for the back-reaction to be added to it. The more general Hilbert space $H(q)$ of functions that obey equation (4) allows for the back-reaction to be taken into account.

In this more general Hilbert space we have introduced the winding operator $\hat{w}$, the flux operator $\hat{s}$, and the w–s phase space, which is a discretized torus $Z(q) \times Z(q)$. This phase space describes the global properties of the system. Indeed, w is a winding number defining the position of the charge in the covering space of the circle $(x + 2\pi w)$. s is magnetic flux and through the non-trivial boundary conditions also appears in the momentum $N + (s/q)$. It is clear that both quantities describe global properties.

In this phase space we have studied displacement operators and the corresponding Heisenberg–Weyl group, and the operators of equation (72) and the corresponding $SL(2, Z(q))$ group. The operators (72) lead to the w' and s' of equations (81) and (82) which are nicely interpreted by the fact that changes in w are intimately related with changes in s (and this leads to w'), and changes in s are intimately related with changes in w (and this leads to s'). This is precisely the inclusion of the back-reaction.

These transformations describe coherent evolution of the system. By that we mean that, at least stroboscopically, the operators performing the evolution of the system form a group. The corresponding Hamiltonians are quadratic Hamiltonians with coefficients that take discrete values. They describe the full interaction between the electric charge and the magnetic flux, going beyond the external field approximation and taking into account the back-reaction of the electric charge on the magnetic flux. Understanding of the coherent evolution is essential for a further study of more complicated types of evolution in these systems (e.g. at times between the stroboscopic ones, or with other more complicated Hamiltonians, etc).

The fact that s–w are dual quantum variables implies that there exists an uncertainty principle. Qualitatively it says that if Δs is small then Δw is large, and *vice versa*. Quantitatively the uncertainty principle in finite systems is expressed with the so-called entropic uncertainty relations (e.g. [13] and references therein). It is interesting to point out that the two extreme limits of the uncertainty principle ($\Delta s = 0$ and $\Delta w = 0$) correspond to the two extreme cases that we have discussed in section 3. When $\Delta s = 0$ we have the Aharanov–Bohm type of experiment in which the value of the flux is well defined and the winding number has large uncertainty. When $\Delta w = 0$ we have the 'which-path' type of experiment in which the path of the particle has a well-defined winding number and the flux has large uncertainty. In intermediate situations both the winding number and the flux have small uncertainties Δw and Δs, correspondingly.

As a final comment we point out that [21] have considered in a mathematical context the case where $\theta/2\pi$ is an irrational number. Also [22] have studied the case where ϕ is a time-dependent, non-classical magnetic flux.

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